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ALTERNATIVE METHOD FOR ASYMPTOTIC FORMULA
OF ADIABATIC RATIO

(Formula for Adiabatic RF Manipulation)

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Alternative Method for Asymptotic Formula of Adiabatic Ratio

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Considerations about the motion of the Courant and Snyder Invariant curve [1] are presented which lead to an exact formula for defining the adiabaticity of a linear harmonic oscillator, described in terms of the auxiliary function for the invariant [2-5]. For a typical example in which an oscillation frequency changes at a finite rate and in a finite period, an asymptotic expression of this formula designated as the adiabatic ratio is obtained by a systematic procedure. This systematic procedure involves a method of asymptotically solving the non-linear auxiliary equation.

1. Introduction. We shall consider a time-dependent harmonic oscillator described by the Hamiltonian

$$H(x, p; t) = (1/2) \left[p^2 + \lambda(t) x^2 \right] \quad (1)$$

where $\lambda(t)$ is the time-varying parameter. It is well known that for sufficiently slow change in the parameter the action variable of system (1) is an approximate constant, which we designate as an "adiabatic constant". The proof for adiabatic invariance of the "action integral" has been given, for example, in the literature [6]. Unfortunately, the adiabatic theorem does not tell quantitatively how slow the change in the parameter must be for the adiabatic theorem to hold. Practically, we would expect to calculate the increase in the action integral when the change in the Hamiltonian is nearly adiabatic and even abrupt. So far approximate methods to calculate the increase have been presented by several authors [7-10] and in particular applied to the following two cases:

$$\lambda(t) = \left[\omega_{\infty} - (\omega_{\infty} - \omega_0) \exp(-\beta^2 t^2) \right]^2$$

where ω_{∞} is the asymptotic limit $t \rightarrow \pm\infty$, and ω_0 is the value at $t=0$,

$$\lambda(t) = \omega_0^2 \left[\frac{1}{2} (n+1) + \frac{1}{2} (n-1) \tanh(\alpha t) \right]^2$$

where n is an integer.

These calculating methods require a finite varying ratio and an infinite varying period of the parameter $\lambda(t)$. The later requirement is not always compatible with real situations. In addition, although it is not immovable, intricacies of an initial

phase problem is unavoidable in them. In this paper, a method for calculating the maximum increase of the action integral for the case where the parameter changes at the finite varying rate and in a finite varying period is presented. In this method, an initial phase problem doesn't appear inherently.

We point out that the system (1) has an exact dynamical invariant independent of the change in the parameter, which is designated as the "Courant and Snyder invariant"

$$I(x, p; t) = [1/2\beta(t)] \left\{ x^2 + \left[\frac{1}{2} \dot{\beta}(t) x - \beta(t) p \right]^2 \right\}, \quad (2)$$

where $\beta(t)$ satisfies the auxiliary differential equation

$$\frac{1}{2} \beta \ddot{\beta} - \frac{1}{4} \dot{\beta}^2 + \lambda(t) \beta^2 = 1. \quad (3)$$

When $\lambda(t)$ is constant, the invariant I is exactly identical with the action variable of the system (1) if we choose the initial condition

$$\beta(-\infty) = 1/\sqrt{\lambda}, \quad \dot{\beta}(-\infty) = 0. \quad (4)$$

For a time-varying function $\lambda(t)$, from (2), we know that a sequence of infinite phase points which have a certain constant value of I at an arbitrary time behaves as a deformable moving ellipse in the phase space $(x, p; t)$ after that time. The form of such a ellipse, called an "invariant curve" in the following, is uniquely determined by the auxiliary differential equation (3) alone.

We will now consider the case when the parameter λ changes from a constant value λ_1 to another constant value λ_2 in a finite period. It is noted that such cases often appear in real situations. That is

$$\lambda(t) = \begin{cases} \lambda_1 & (\dot{\lambda}(t_1)=0) \quad \text{for } t \leq t_1, \\ \lambda_2 & (\dot{\lambda}(t_2)=0) \quad \text{for } t \geq t_2. \end{cases} \quad (5)$$

Fig. 1

We consider the invariant curve described in the term

$$I(x, p; t_0) = I_0, \quad t_0 < t_1, \quad (6)$$

with a constant I_0 . The quantity I_0 is equal to the value of the action variable of infinite phase points which comprise the invariant curve, as mentioned above. So if we continue to examine the motion of the invariant curve at $t > t_2$, we find the exact time-evolution of such a sequence of infinite points which once possessed the same action variable J . This fact will give us useful informations about a change in the action variable. Furthermore, it may enable us to estimate the adiabaticity of the system (1). In fact, Symon [11] has derived the maximum and minimum change of an action integral, based on the similar idea. Here, from a geometrical point of view, we shall derive an exact formula for estimating the adiabaticity, which is equivalent to that of Symon. This is designated as the adiabatic ratio in the following. For the case stated in the early part of this section, an asymptotic expression of this adiabatic ratio shall be derived analytically, by using an asymptotic solution of the auxiliary equation (3). The analytical results are found to be in agreement with a numerical evaluation.

2. Motion of invariant curve and adiabatic ratio. We may characterize an ellipse or an invariant curve by two parameters of ξ, δ which are a function of $\beta(t)$ and $\dot{\beta}(t)$ (See Fig.2)

Fig. 2

If we choose the initial conditions $\beta(t_0)=1/\sqrt{\lambda_1}$, $\dot{\beta}(t_0)=0$, the solution of (3) is

$$\beta(t) = 1/\sqrt{\lambda_1}, \quad \dot{\beta}(t) = 0, \quad t_0 \leq t \leq t_1. \quad (7)$$

Therefore, before the variation of the Hamiltonian the form of the above ellipse remains unchanged and its motion is only parallel displacement along the axis of time. At $t=t_1$, the ellipse begins to move, following the time-evolution of $\beta(t)$ which is determined by Eq.(3). After the variation of the Hamiltonian, i.e., $t \geq t_2$, the ellipse continues to move unless $(\beta(t_2), \dot{\beta}(t_2))$ is equal to $(1/\sqrt{\lambda_2}, 0)$ (See Fig.2).

Now we write the phase space area surrounded by the invariant curve which remains constant in the region $t_0 \leq t < \infty$, in the term

$$S_0 = 2\pi I_0. \quad (8)$$

In addition, we shall define S as the cross section of the outside envelope which the moving ellipse makes after $t=t_2$ (See Fig.3).

Fig. 3

It is noted that the phase area between the outside and inside envelopes is the area effectively occupied by the phase points which have the same action variable J at $t \leq t_1$. From Fig.3, the cross section S is described in the form

$$S = \pi \max \xi(t) \max \delta(t), \quad t \geq t_2. \quad (9)$$

Then it is trivial to write $\xi(t)$, $\delta(t)$ in the terms of $\beta(t)$, $\dot{\beta}(t)$

$$\xi(t) = \sqrt{2I_0 \beta(t)}, \quad (10a)$$

$$\delta(t) = \sqrt{2I_0 \gamma(t)}, \quad (10b)$$

where

$$\gamma(t) = [1 + \dot{\beta}(t)^2 / 4] / \beta(t). \quad (11)$$

Here we define the adiabatic ratio r in the term

$$r = \frac{S}{S_0} \quad \text{or} \quad [\max \beta(t) \max \gamma(t)]^{1/2}. \quad (12)$$

The solution of (3) with the constant $\lambda = \lambda_2$ is well known:

$$\beta(t) = \sqrt{A^2 + B^2 + 1/\lambda_2} + A \cos[2\sqrt{\lambda_2}(t-t_2)] + B \sin[2\sqrt{\lambda_2}(t-t_2)], \quad (13a)$$

$$\dot{\beta}(t) = 2\sqrt{\lambda_2} \{ -A \sin[2\sqrt{\lambda_2}(t-t_2)] + B \cos[2\sqrt{\lambda_2}(t-t_2)] \}, \quad (13b)$$

where A and B satisfy the boundary condition

$$\beta(t_2) = \sqrt{A^2 + B^2 + 1/\lambda_2} + A, \quad (14a)$$

$$\dot{\beta}(t_2) = 2\sqrt{\lambda_2} B \quad (14b)$$

From (13a) and (13b), it is trivial to obtain the maximum values of $\beta(t)$ and $\gamma(t)$. We write these values in the terms

$$\text{Max } \beta(t) = \sqrt{A^2 + B^2 + 1/\lambda_2} + \sqrt{A^2 + B^2}, \quad (15a)$$

$$\text{Max } \gamma(t) = 1 / \left(\sqrt{A^2 + B^2 + 1/\lambda_2} - \sqrt{A^2 + B^2} \right). \quad (15b)$$

Substituting (15a) and (15b) into (12), we obtain

$$r = \sqrt{\lambda_2(A^2 + B^2) + 1} + \sqrt{\lambda_2(A^2 + B^2)}. \quad (16)$$

This agrees to the result obtained by Symon [11], though notations of terms are slightly different. Replace the term of $A^2 + B^2$ by the auxiliary function and its time-derivative. So that (16) becomes

$$r = X + \sqrt{X^2 - 1}, \quad (17)$$

where

$$X = \frac{\sqrt{\lambda_2}}{2} \left[\beta(t_2) + \frac{\dot{\beta}^2(t_2)}{4\lambda_2\beta(t_2)} + \frac{1}{\lambda_2\beta(t_2)} \right]. \quad (18)$$

According to Eq. (18), it is identified that only the values of the auxiliary function and its time-derivative at $t=t_2$ are required in order to obtain an exact expression of the adiabatic ratio for a certain given change in the parameter as seen in Fig. 1. In the next section, we shall discuss how these values can be evaluated.

3. Approximate Calculation Method of Adiabatic Ratio We shall consider the cosine-like change of the coefficient which often appears in beam dynamics of accelerators and storage rings, and other fields of physics:

$$\lambda(t) = \left\{ \frac{1}{2} \left[\omega_1 + \omega_2 + (\omega_1 - \omega_2) \cos \frac{\pi}{T} (t - t_1) \right] \right\}^2 \quad (19)$$

$t_1 \leq t \leq t_2$

where $\lambda_1 = \omega_1^2$, $\lambda_2 = \omega_2^2$ and $T(=t_2-t_1)$ is a finite transition time. Introduce the parameter defined as a ratio of the transition time to the initial oscillation period

$$\mu = T/T_1 \quad (T_1 = 2\pi/\omega_1), \quad (20a)$$

and a decrement ratio of the oscillation frequency, that is,

$$\epsilon = 1 - \frac{\omega_2}{\omega_1}. \quad (20b)$$

Furthermore, making a change of time-scale

$$\theta = \pi(t-t_1)/T, \quad (20c)$$

we have the normalized equation of motion

$$\frac{d^2x}{d\theta^2} + \mu^2(2-\epsilon + \epsilon \cos \theta)^2 x = 0. \quad (21)$$

Here, for a given ϵ , the parameter μ can be regarded as an adiabatic parameter, since largeness μ is a measure of the slowness of the coefficient $\lambda(t)$. Then, introducing the root square function of the auxiliary function

$$p(\theta) = [\beta(\theta)]^{1/2},$$

we can reduce the auxiliary equation (3) to the following non-linear equation:

$$\ddot{p} + R f(\theta) p = \frac{1}{p^3}, \quad (\dot{\cdot} \equiv \frac{d}{d\theta}) \quad (22)$$

with the initial condition $p(0) = [R f(0)]^{-1/4}$ and $\dot{p}(0) = 0$,

where

$$R = \mu^2(2-\epsilon)^2 \quad (23a)$$

$$f(\theta) = \left(1 + \frac{\epsilon}{2-\epsilon} \cos \theta \right)^2. \quad (23b)$$

To solve Eq.(22), let us determine a straightforward asymptotic expansion for \mathcal{R}^{-1} . Thus we assume that

$$p(\theta) = p_0(\theta) + \frac{1}{\mathcal{R}} p_1(\theta) + \frac{1}{\mathcal{R}^2} p_2(\theta) + \dots \quad (24)$$

Substituting (24) into

$$p^3 [\ddot{p} + \mathcal{R} f(\theta) p] - 1 = 0, \quad (25)$$

and equating coefficients of equal powers of \mathcal{R}^{-1} to zero lead to

$$\mathcal{R} f(\theta) p_0^4 - 1 = 0, \quad (26a)$$

$$p_0^3 [\ddot{p}_0 + f(\theta) p_1] + 3 f(\theta) p_0^2 p_1 = 0, \quad (26b)$$

$$p_0^3 [\ddot{p}_1 + f(\theta) p_2] + 3 p_0^2 p_1 [\ddot{p}_0 + f(\theta) p_1] + 3 f(\theta) p_0^2 (p_1^2 + p_0 p_2) = 0, \quad (26c)$$

$$\vdots$$

Here the equal or less terms than the 0-th power of \mathcal{R}^{-1} are included in (26a). The solution of this zeroth-order equation is

$$p_0 = [\mathcal{R} f(\theta)]^{-1/4}. \quad (27a)$$

Further, we can determine the higher approximations by simple algebraic calculations. Substituting (27a) into (26b) and solving for

p_1 , we have

$$\begin{aligned} p_1 &= - \frac{\ddot{p}_0}{4 f(\theta)}, \\ &= \frac{\mathcal{R}^{-1/4}}{16} \left(-\frac{5}{4} f^{-1/4} \dot{f}^2 + f^{-9/4} \ddot{f} \right). \end{aligned} \quad (27b)$$

Substituting (27a) and (27b) into (26c) and solving for p_2 , we have

$$P_2 = - \frac{P_0 \ddot{P}_1 + 3P_1 \ddot{P}_0 + 6f P_1^2}{4f P_0} \quad (27c)$$

Therefore

$$P = R^{-1/4} f^{-1/4} + \frac{1}{16} R^{-5/4} \left(-\frac{5}{4} f^{-13/4} \dot{f}^2 + f^{-1/4} \ddot{f} \right) + O(R^{-9/4}) \quad (28)$$

Though the initial conditions of (22) require

$$\begin{aligned} P_1(0) = P_2(0) &= - - - - - = 0, \\ \dot{P}_1(0) = \dot{P}_2(0) &= - - - - - = 0. \end{aligned}$$

unfortunately, P_1 does not at least satisfy them because of

$$\ddot{f}(\theta) = -\frac{2\epsilon}{2-\epsilon} \left(-\frac{\epsilon}{2-\epsilon} \sin^2 \theta + \frac{\epsilon}{2-\epsilon} (\cos^2 \theta + \cos \theta) \right) \neq 0 \text{ at } \theta=0.$$

As, however, only the values of P , \dot{P} at $\theta = \pi$ are required, we will ignore this fact. For the fast convergency of Eq.(28), we require the following conditions:

$$R \ll 1 \quad \text{or} \quad \mu \gg \frac{1}{2-\epsilon}, \quad (29a)$$

$$\frac{\epsilon}{2-\epsilon} < 1 \quad \text{or} \quad \epsilon < 1. \quad (29b)$$

For a value of μ satisfying the condition (29a), Eq.(28) is a good asymptotic expression, provided that a given value of ϵ is smaller than unity. Now, using the following terms:

$$\begin{aligned} R^{-1/4} &= \mu^{-1/2} (2-\epsilon)^{-1/2}, \\ f(\pi) &= 2^2 \left(\frac{1-\epsilon}{2-\epsilon} \right)^2, & \dot{f}(\pi) &= 0, \\ \ddot{f}(\pi) &= - \frac{2^2 \epsilon (\epsilon-1)}{(2-\epsilon)^2}, \end{aligned}$$

we obtain the asymptotic values of $P(\pi)$ and $\dot{P}(\pi)$

$$P(\pi) = 2^{-1/2} (1-\epsilon)^{-1/2} \mu^{-1/2} + 2^{-13/2} \epsilon (1-\epsilon)^{-7/2} \mu^{-5/2} + O(\mu^{-9/2}) \quad (30a)$$

$$\dot{\beta}(\pi) = 0, \quad (30b)$$

Accordingly, the asymptotic values of the auxiliary function and its time-derivative for $\mu \rightarrow \infty$ become

$$\beta(\pi) = 2^{-1} (1-\epsilon)^{-1} \mu^{-1} + 2^{-6} \epsilon (1-\epsilon)^{-4} \mu^{-3} + O(\mu^{-5}), \quad (31a)$$

$$\dot{\beta}(\pi) = O(\mu^{-5}) \quad (31b)$$

Finally, substituting (31a) and (31b) into (17) or (18), we can write the adiabatic ratio in the form of an asymptotic expansion of the adiabatic parameter μ :

$$r = 1 + 2^{-5} \epsilon (1-\epsilon)^{-3} \mu^{-2} + O(\mu^{-4}) \quad (32)$$

For two examples of $\epsilon = 1/2$ and $2/3$ which mean oscillation frequency reductions of a factor of 2 and 3, the adiabatic ratio is plotted as a function of the adiabatic parameter μ in Fig.4a and 4b.

Fig.4a and 4b

Here, calculations are performed to first order in Eq.(32). In addition, numerically computed values are given in the same figures. They are obtained by substituting the values of $\beta(\pi)$ and $\dot{\beta}(\pi)$ calculated numerically into the exact formula (18). Both results for each case agree well as μ becomes larger.

The present systematic procedure to calculate the values of the auxiliary function and its time-derivative at the just time when the parameter $\lambda(t)$ completes its changing can be applied to other cases

that this parameter smoothly changes in a finite period. Further extensive applications of the present method and a problem connected with the convergency of Eq.(28) not discussed in detail will be given elsewhere.

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References

- [1] E.D.Courant and H.S.Snyder, Ann. Phys. 3 (1958) 1.
- [2] H.R.Lewis and W.B.Riesenfeld, J. Math. Phys. 10 (1969) 1458.
- [3] M.Lutzky, Phys. Lett. 68A (1978) 3.
- [4] J.R.Ray and J.L.Reid, Phys. Lett. 71A (1979) 317.
- [5] H.J.Korsh, Phys. Lett. 74A (1979) 294.
- [6] A.J.Lichtenberg, in: Phase-Space Dynamics of Particles, (John Wiley & Sons, Inc) (1969) 53.
- [7] F.Hertweck and A.Schluter, Z. Naturforsch 12A (1957) 844.
- [8] V.G.Backus, G.A.Lenard and R.Kulsrud, Z. Naturforsch 15A (1960) 1007.
- [9] P.O.Vandervoort, Ann. Phys. 12 (1961) 436.
- [10] J.E.Howard, Phys. Fluids 13 (1970) 2407.
- [11] K.R.Symon, J. Math. Phys. 11 (1970) 1320.

Figure Captions

Fig.1 Time-varying coefficient $\lambda(t)$

Fig.2 Time-evolution of invariant curve
characteristic parameters ξ and δ .

Fig.3 Outside envelope and inside envelope of moving invariant curve

Fig.4a Adiabatic ratio versus the adiabatic parameter μ for $\epsilon = 1/2$.

The dots are numerically computed values and the line is the
theoretical prediction.

Fig.4b Adiabatic ratio versus the adiabatic parameter μ for $\epsilon = 2/3$.

The dots are numerically computed values and the line is the
theoretical prediction.

Fig. 1

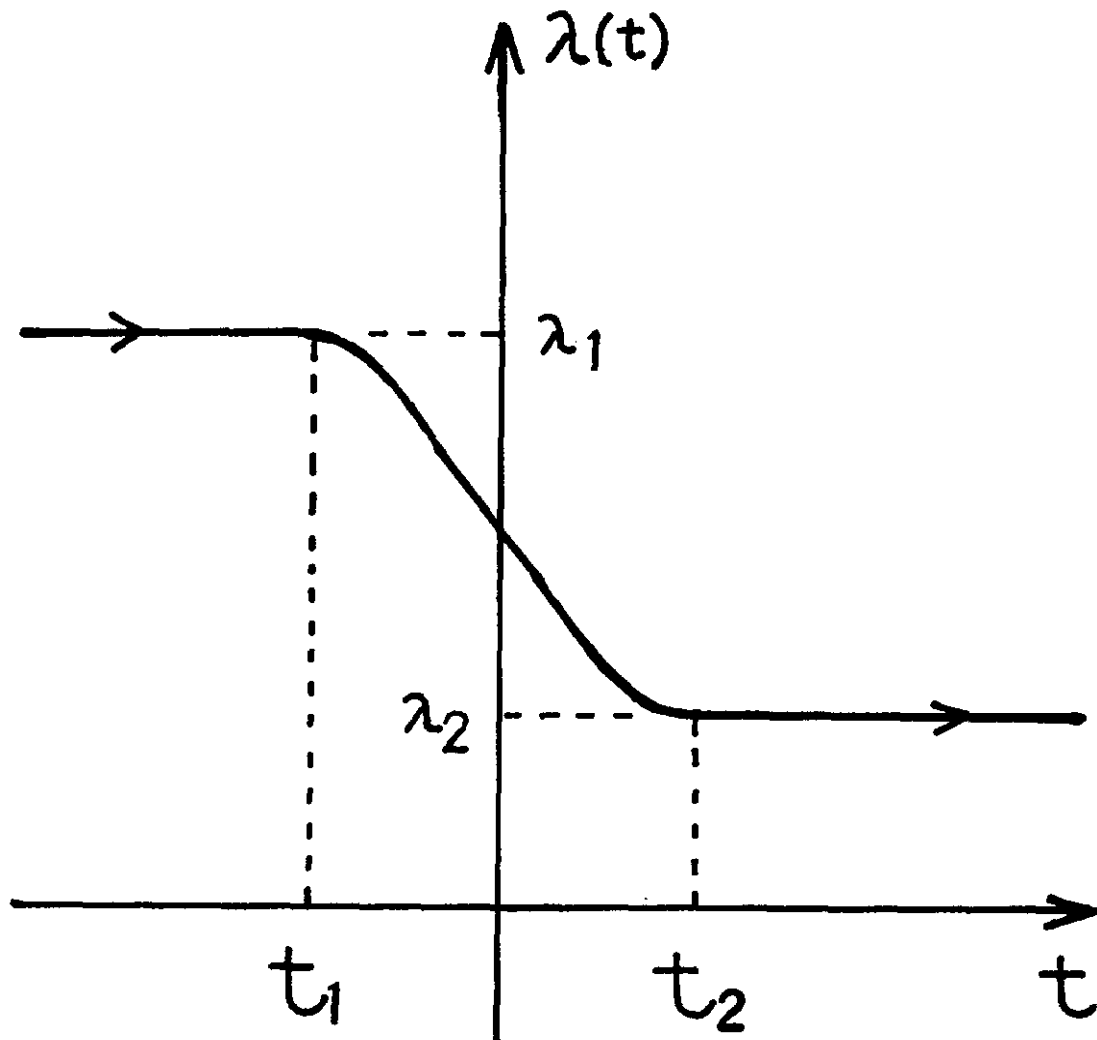


Fig. 2

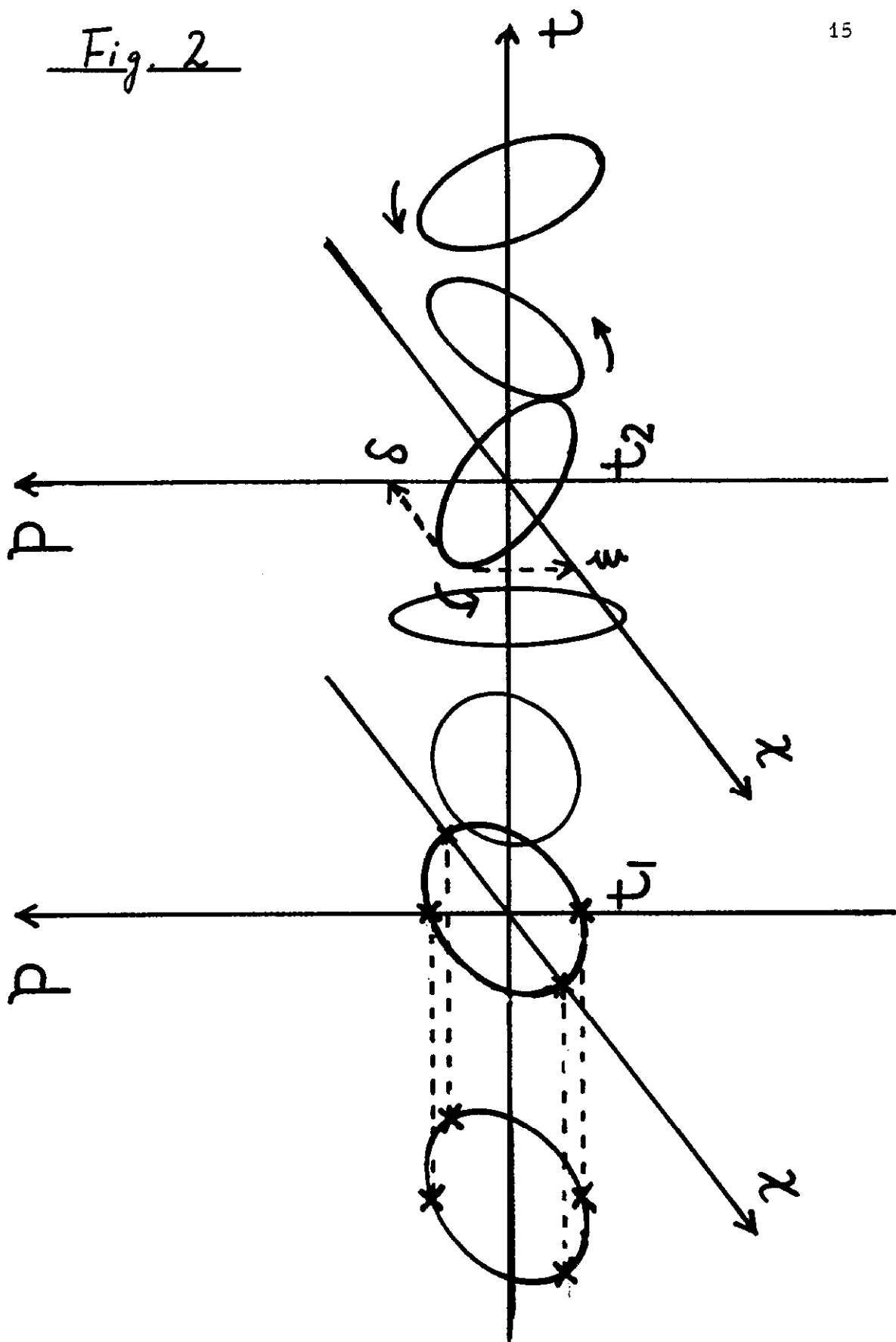


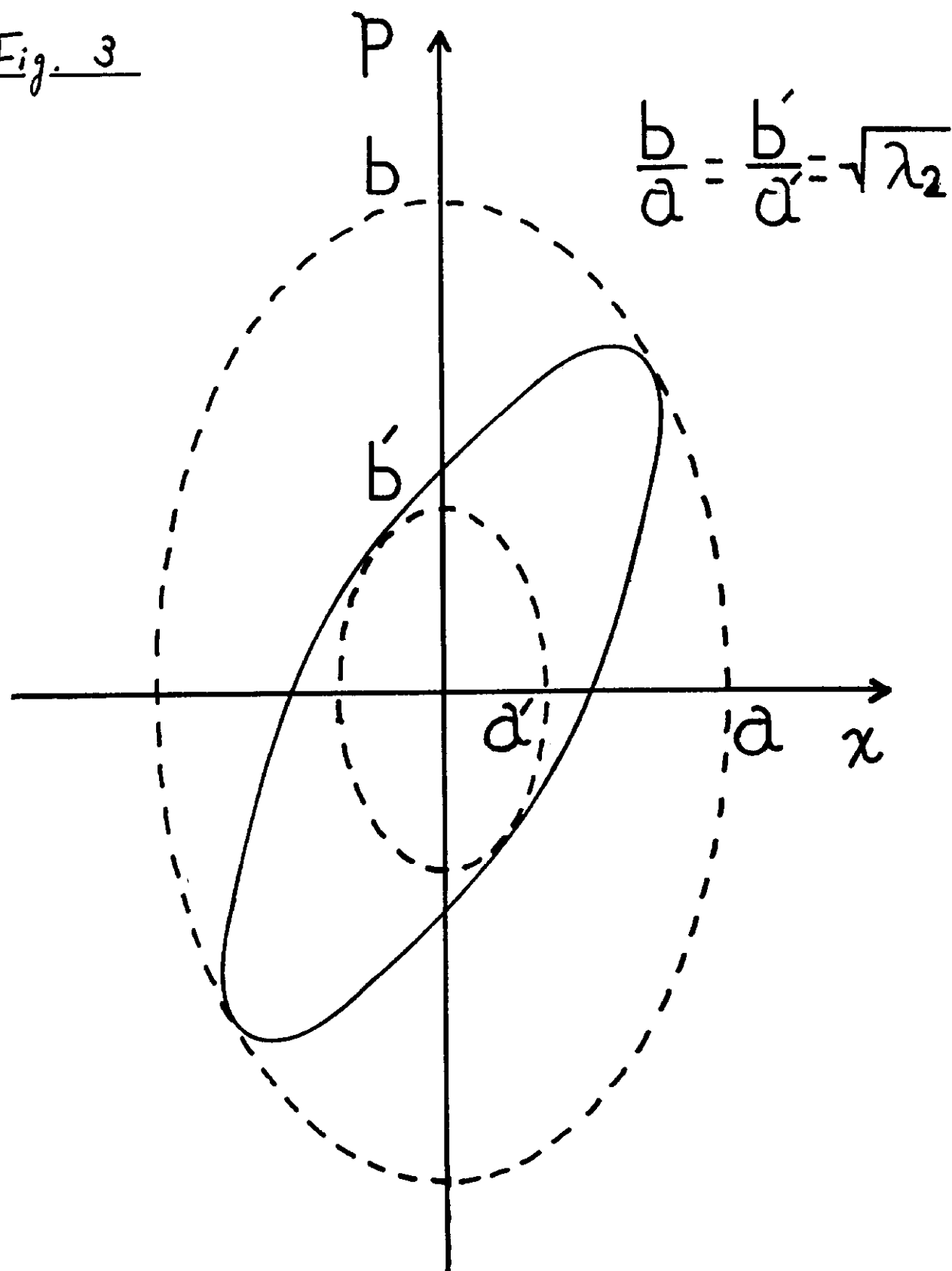
Fig. 3

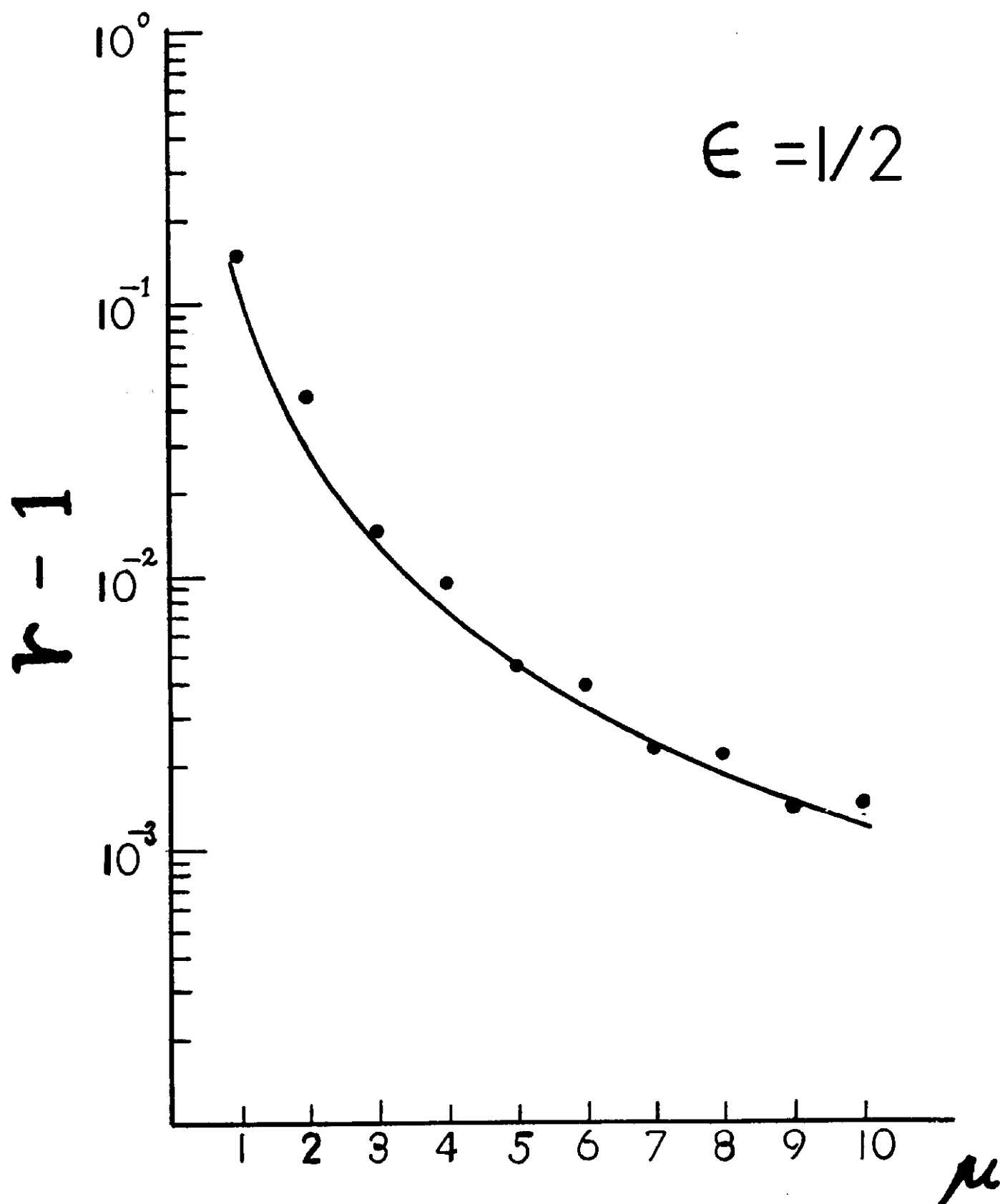
Fig. 4a

Fig. 4b

$$\epsilon = 2/3$$

